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THE CONE TOPOLOGY ON A MANIFOLD WITHOUT FOCAL POINTS

M. S. GOTO

Introduction

Let M be a complete, simply connected Riemannian manifold without focal points. Let $\alpha(t)$ and $\beta(t)$, $t \geqslant 0$, be geodesic rays parametrized by their arc lengths, respectively. Then α and β are asymptotic if the distance between $\alpha(t)$ and $\beta(t)$ is bounded for all $t \geqslant 0$. Let $M(\infty)$ be the set of all classes of asymptotic geodesic rays and let $\overline{M} = M \cup M(\infty)$. In [4] it was proved that for any point p in M and a geodesic ray α, there exists a unique geodesic ray β asymptotic to α with $\beta(0) = p$.

Let E be $\mathbf{R}^{n+1}$ with the natural euclidean metric. Then E is an example of M. In this case two geodesic rays $\alpha(t) = a + tv(\|v\| = 1)$ and $\beta(t) = b + tw(\|w\| = 1)$ are asymptotic if and only if they are parallel, i.e., $v = w$. We denote the asymptotic class containing α by ∞v, and suppose that the ray is extended to the interval $[0, \infty]$ by putting $\alpha(\infty) = \infty v$. Then $E(\infty)$ has the natural topology as the unit sphere S^n, and $\overline{E}$ can be identified with the closed unit $(n + 1)$ – disk.

The purpose of this note is to prove the following:

Theorem. *Let M be a complete, simply connected Riemannian manifold without focal points. Then $\overline{M}$ has a canonical topology with the following property: For any $p \in M$, the exponential map: $T_pM \to M$ extends uniquely to a homeomorphism from $\overline{T_pM}$ onto $\overline{M}$.*

The topology is called the *cone topology* since for each point x in $M(\infty)$, cones containing x form a local basis at x.

The theorem is known in the case of nonpositive curvature (see [2]). In the case of no focal points, it was proved if either the dimension of M is 2, or the geodesic flow of M is of Anosov type (see [4]). The proof here refers to [3] and [4].

Proof of the theorem. Let $K(t)$ be a symmetric $n \times n$ matrix valued continuous function defined for all $t \in \mathbf{R}$, and consider the $n \times n$ matrix

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differential equation

$$X''(t) + K(t)X(t) = 0, \tag{J}$$

where the derivatives are taken componentwise. Let A be the solution of (J) with the initial conditions $A(0) = 0$ and $A'(0) = I$ (the identity matrix). Also for $s > 0$ let D_s be the solution with the boundary conditions $D_s(0) = I$ and $D_s(s) = 0$. Then it is known that $\lim_{s\to\infty} D_s = D$ exists and is given by

$$D(t) = A(t)\int_t^{\infty}(A^*A)^{-1}(u)\,du,$$

where A^* denotes the transposed matrix of A.

Hereafter, M denotes a complete, simply connected Riemannian manifold of dimension $n + 1$ and class C^{∞} without focal points. For $p \in M$, let T_pM denote the tangent space at p, and let $S_pM = \{v \in T_pM;\ \|v\| = 1\}$. Let SM be the unit tangent bundle. For $v \in S_pM$ we denote by γ_v the geodesic ray with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, parametrized by its arc length. Let $\{e_1(t), \ldots, e_n(t), e_{n+1}(t) = \gamma_v'(t)\}$ be a parallel orthonormal frame field along the geodesic γ_v. If $Y(t) = \sum_{i=1}^{n} y_i(t)e_i(t)$ is a normal vector along γ_v, then we can identify Y with the curve $t \mapsto (y_1(t), \ldots, y_n(t))$ in $\mathbf{R}^n$. For each $t \in \mathbf{R}$ we denote $K(t) = (\langle R(e_i(t), \gamma'(t))\gamma'(t), e_j(t)\rangle)$, where R is the curvature tensor, and consider (J) for this $K(t)$. The solution given above will be denoted by D_v.

Next, we define a map $b_{vs}\colon M \to \mathbf{R}$ for $v \in SM$ by

$$b_{vs}(q) = s - d(\gamma_v(s), q),$$

where d denotes the distance. Then $\lim_{s\to\infty} b_{vs} = b_v$ exists. The function b_v is called the *Busemann function* with respect to v, and is known to be of class C^2.

Let v be in SM, and $q \in M$. Then there exists a unique geodesic ray starting at q asymptotic to γ_v, and the tangent vector of the geodesic ray at q is given by $(\nabla b_v)(q)$. To prove our theorem, it is enough to see the continuous dependence of ∇b_v on the parameter v according to the discussion in [2, §2].

Let p be a point of M, and v a unit vector at p. Then $D_v'(0)$ is a linear transformation of the vector space $v^{\perp} = \{x \in T_pM;\ x \perp v\}$. We shall consider the vector bundle over SM given by

$$\{(v, \varphi);\ v \in SM, \varphi \in \operatorname{End}(v^{\perp})\},$$

and the cross section: $v \mapsto D_v'(0)$. In [3] Eschenburg obtained that

$$\nabla_w(\nabla b_v) = D_v'(0)(w) \quad \text{for } w \in v^{\perp},$$

and that $D_v'(0)$ depends continuously on v.

We shall now extend $D_v'(0)$, $v \in SM$, to an endomorphism $\mathfrak{D}(v)$ of T_pM by putting

$$\begin{cases} \mathfrak{D}(v)(w) = D_v'(0)(w) & \text{for } w \in v^\perp, \\ \mathfrak{D}(v)(v) = 0. \end{cases}$$

Then $\mathfrak{D}(v)$ is a cross section of the vector bundle

$$\{(v, \psi);\ v \in S_pM, \psi \in \mathrm{End}(T_pM) \text{ for } p \in M\}$$

over SM and is obviously continuous. On the other hand,

$$\nabla_v(\nabla b_v) = 0,$$

and hence

$$(*) \qquad \nabla(\nabla b_v) = \mathfrak{D}(v)$$

is continuous with respect to $v \in SM$.

Let p and q be distinct ponts in M. We pick a smooth curve $\sigma(s)$ such that $\sigma(0) = p$ and $\sigma(1) = q$, and shall consider a differential equation

$$(**) \qquad \frac{\nabla}{ds} X(s) = \mathfrak{D}(X(s))(\sigma'(s)),$$

where $X(s)$ is a unit vector field along $\sigma(s)$ of class C^1. For a unit vector v at p,

$$Y_v(s) := (\nabla b_v)(\sigma(s))$$

is a solution of $(**)$ with $Y_v(0) = v$. We shall prove that $Y_v(s)$ is the unique solution with the initial condition v.

Suppose that $X(s)$ is a solution of $(**)$ with $X(0) = v$. We consider the variation $f(t, s) = \exp_{\sigma(s)} tX(s)$, $s \in [0, 1]$, $t \geqslant 0$, of the geodesic ray γ_v. Then $J_s(t) := (\partial/\partial s) f(t, s)$ is a Jacobi field for every s. Since $X(s)$ is of class C^1, $J_s(t)$ is continuous with respect to s. Fix $s_0 \in [0, 1]$ and put $w = X(s_0)$. Then

$$Y_w(s) := \nabla b_w(\sigma(s))$$

is a solution of $(**)$ with $Y_w(s_0) = w$. We put $\tilde{f}(t, s) = \exp_{\sigma(s)} tY_w(s)$ and $\tilde{J}(t) = (\partial/\partial s)\tilde{f}(t, s)|_{s=s_0}$. Then $\tilde{J}(t)$ is the Jacobi field along γ_w with

$$\tilde{J}(0) = \sigma'(s_0), \tilde{J}'(0) = \mathfrak{D}(w)(\sigma'(s_0)).$$

Moreover, since the variational curves $t \mapsto \tilde{f}(t, s)$ are all asymptotic to γ_w, it follows that

$$\|\tilde{J}(t)\| \leqslant \|\tilde{J}(0)\| \quad \text{for any } t \geqslant 0.$$

On the other hand, $J_{s_0}(0) = \sigma'(s_0)$ and $J_{s_0}'(0) = \mathfrak{D}(w)(\sigma'(s_0))$. Hence the Jacobi field J_{s_0} coincides with $\tilde{J}$. Thus

$$\|J_s(t)\| \leqslant \|J_s(0)\| = \|\sigma'(s)\| \quad \text{for } s \in [0, 1], t \geqslant 0.$$

Therefore

$$d(\gamma_v(t), f(t, s_0)) \leqslant \int_0^{s_0} \|J_s(t)\|\, ds \leqslant \int_0^{s_0} \|\sigma'(s)\|\, ds,$$

and hence the geodesic ray $t \mapsto f(t, s) = \exp_{\sigma(s)} tX(s)$ is asymptotic to γ_v for any $s \in [0, 1]$. By the uniqueness of asymptotic geodesic rays, we have

$$X(s) = (\nabla b_v)(\sigma(s)).$$

Thus the equation (**) has a unique solution. Because of the continuity of $\mathfrak{D}$, the solution of (**) depends continuously on the initial value by a theorem of differential equations (cf. [1, Chapter 2, Theorem 4.1]). Namely, ∇b_v is continuous with respect to v. Hence the proof is complete.

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INSTITUTE FOR ADVANCED STUDY, PRINCETON